The Spline-Threshold-GARCH Volatility Model and Tail Risk

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Abstract

We introduce an asymmetric Spline-Threshold-GARCH model that generalizes the Spline-GARCH and asymmetric TARCH models. Monte Carlo experiments of this model and applications for SPX and DJI equity indices show that our model has better fit and has higher persistence for the GARCH parameters when the returns are negative. We suggest to use our model for forecasting volatility and tail risk measures as it is more general and robust than Spline-GARCH or Spline-TARCH models.

Key Words: Spline, Threshold GARCH, Tail Risk.

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1. Introduction

The generalized autoregressive conditional heteroscedasticity (GARCH) model is one of the most popular volatility models used by financial practitioners and academics. Since its introduction there have been many extensions of GARCH models that resulted in better statistical fit and forecasts. For example, TARCH or GJR-GARCH (Glosten, Jagannathan, & Runkle (1993)) is one of the well-known extensions of GARCH models with an asymmetric term which captures the effect of negative shocks in equity prices on volatility commonly referred to as a "leverage" effect. EGARCH introduced by Nelson (1991) is an alternative asymmetric model of the logarithmic transformation of conditional variance that does not require positivity constraints on parameters. Different volatility regimes can be captured by Markov Regime Switching ARCH and GARCH models allowing for stochastic time variation in parameters. These models were introduced by Cai (1994) and Hamilton and Susmel (1994) correspondingly.

Since tail risk measures use forecasts of volatility model specification is essential for risk management. Engle and Mezrich (1995) introduced a way to estimate value at risk (VaR) using a GARCH model, while Hull and White (1998) proved that a GARCH model has a better performance than a stochastic volatility model in calculation of VaR. The TARCH or GJR-GARCH model was used by Brownlees and Engle (2012) among others for forecasting volatility and measurement of tail and systemic risks.

The typical feature of the GARCH family models is that the long run volatility forecast converges to a constant level. One exception is the Spline-GARCH model of Engle and Rangel (2008) that allows the unconditional variance to change with time as an exponential spline and the high frequency component to be represented by a unit GARCH process. This model may incorporate macroeconomic and financial variables into the slow moving component and as shown in Engle and Rangel (2008) improves long run forecasts of international equity indices. A special feature of this model is that the unconditional volatility coincides with the low-frequency volatility. However, this model lacks a well-documented asymmetry in volatility.

There were very few other applications of Spline-GARCH models in the literature. The Factor-Spline-GARCH model developed in Rangel and Engle (2012) is used to estimate high and low frequency components of equity correlations. Their model is a combination of the asymmetric Spline GJR-GARCH (or TARCH) and the DCC (dynamic conditional correlations) models. Another application of an asymmetric Spline GJR-GARCH model for commodity volatilities is done in Carpantier and Dufays (2012).

In this paper we generalize the asymmetric Spline-GARCH models by combining the Spline-GARCH model and a more general threshold GARCH model introduced in Goldman (2012). The widely used asymmetric GJR-GARCH (TARCH) model has a problem that the unconstrained estimated coefficient of α often has a negative value for equity indices. A typical solution to this problem is setting the coefficient of α to zero in the constrained Maximum Likelihood optimization. Following Goldman (2012) we use a generalized threshold GARCH (GTARCH) model where both coefficients, α and β, in the GARCH model are allowed to change to reflect the asymmetry of volatility due to negative shocks. We show that the GTARCH model fits better as well as does not have a negative alpha bias for several equity indices and numerical examples.

This paper is organized as follows. In the following section we review the Spline-GARCH models and introduce our new Spline-Threshold-GARCH (Spline-GTARCH) model. In section three, we estimate models using historical S&P500 and DJ30 data and measure value at risk VaR and expected shortfall ES for each model. In section four we run Monte Carlo experiments. The last section provides conclusion and future research.

2. Spline-GARCH Volatility Models

Consider time series of returns r_t. The Engle and Rangel (2008) Spline-GARCH model is given by the following GARCH variance g_t and quadratic spline τ_t:

$$r_t - E_{t-1} r_t = \sqrt{\tau_t g_t}\, z_t \tag{1}$$

$$g_t = (1-\alpha-\beta) + \alpha\left(\frac{(r_{t-1} - E_{t-2} r_{t-1})^2}{\tau_{t-1}}\right) + \beta\, \mathrm{g}_{t-1} \tag{2}$$

$$\tau_t = c \exp\left(\omega_0 t + \sum_{i=1}^{k} \omega_i ((t - t_{i-1})_+)^2 + m_t \gamma \right) \tag{3}$$

where $(t - t_i)_+ = \begin{cases} (t - t_i) & if t > t_i \\ 0 & otherwise, \end{cases}$

z_t is a standard Gaussian white noise process; m_t is the set of weakly exogenous variables (i.e. macroeconomic variables); $(\mathrm{t}_0 = 0, \mathrm{t}_1, \dots, \mathrm{t}_k = T)$ is a partition of total number of observations T into k equal subintervals. The vector of all jointly estimated parameters in the model is $\varphi = (\alpha, \beta, c, w_0, w_1 \dots w_k)$ The parameters are estimated using Maximum likelihood and the number of knots, k, is chosen by minimizing the Schwarz information criterion (SIC).[1]

Since the constant term in the GARCH variance equation is normalized the long run (unconditional) variance is determined by the spline. Higher number of knots indicates more cycles in the low-frequency volatility, while parameters $(w_0, w_1 \ldots\ w_k)$ represent the sharpness of the cycles.

Since GARCH model is symmetric we need to extend it and allow for negative shocks to increase volatility more than positive shocks. In the literature this asymmetry is referred to the leverage effect and is commonly modeled using GJR-GARCH Model (Glosten et al. (1993)) also called threshold ARCH or TARCH model. The TARCH (1,1, 1) model is given by

$$r_t = \mu + \varepsilon_t \tag{4}$$

$$g_t = \omega + \alpha\varepsilon_{t-1}^2 + \gamma I_{t-1}\varepsilon_{t-1}^2 + \beta g_{t-1}$$

$$\text{where} \quad I_{t-1} = \begin{cases} 0\ if\ r_{t-1} \geq \mu \\ 1\ if\ \ r_{t-1} \leq \mu \end{cases}$$

Here the leverage coefficient (γ) is applied to negative innovations increasing the effect of negative shocks.

However, there is a problem with the threshold ARCH model above since coefficient α may take negative values in practice. In such case a constrained optimization results in α equal to zero. Goldman(2012) suggested the following more general Threshold GARCH or GTARCH (1, 1, 1) model:

$$r_t = \mu + \varepsilon_t \tag{5}$$

$$g_t = \omega + \alpha\varepsilon_{t-1}^2 + \gamma I_{t-1}\varepsilon_{t-1}^2 + \beta g_{t-1} + \delta I_{t-1} g_{t-1}$$

In this model both parameters γ and δ create the asymmetric response of volatility to negative shocks. Results below show that by allowing both ARCH and GARCH parameters to change with negative news results in better statistical fit and smaller SIC.

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[1] or Bayesian information criterion. We estimate the model starting with 1 knot up to maximum k=10 knots.

Moreover, the Threshold-GARCH model not only captures the leverage effect but also shows higher persistence for negative returns compared to a simpler TARCH Model.

The model introduced in this paper is the combined Spline-Threshold GARCH (Spline-GTARCH) model from equations (1)-(3) and (5) given by

$$r_t - E_{t-1} r_t = \sqrt{\tau_t g_t}\, z_t \tag{6}$$

$$\begin{aligned} g_t &= (1-\alpha-\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)+\alpha(\frac{(r_{t-1}-E_{t-2}r_{t-1})^2}{\tau_{t-1}})+\beta\, \mathrm{g}_{t-1} \\ &+\gamma(\frac{(r_{t-1}-E_{t-2}r_{t-1})^2}{\tau_{t-1}})I+\delta\, \mathrm{g}_{t-1}\, I \end{aligned} \tag{7}$$

$$\tau_t = c\exp(\omega_0 t + \sum_{i=1}^{k} \omega_i ((t-t_{i-1})_+)^2 \tag{8}$$

$$where\ I_{t-1} = \begin{cases} 0\ if\ r_{t-1} \geq E_{t-2} r_{t-1} \\ 1\ if\ \ r_{t-1} \leq E_{t-2} r_{t-1} \end{cases}$$

3. Estimation of the Spline-Threshold GARCH Model

We use Maximum Likelihood Estimation method (MLE) to estimate all the parameters $(\alpha, \beta, \gamma, \delta, c, w_0, w_1 \dots w_k)$ simultaneously. The restrictions on the parameters are given by $\alpha, \beta, \gamma, \delta, c > 0$ and $\alpha + \beta + \frac{1}{2}\gamma + \frac{1}{2}\delta < 1$. The assumption that ε has Gaussian distribution is not crucial since asymptotically a quasi- maximum likelihood approach can be used if returns are not Gaussian.

The likelihood function is the product of:

$$f(r_t, \mathrm{E}_{t-1} r_t, g_t, \tau_t) = \frac{1}{\tau_t \sqrt{2\pi g}_t} e^{-\frac{1}{2g_t}(\frac{r_t - E_{t-1} r_t}{\tau_t})^2} \tag{9}$$

3.1. Monte Carlo Experiments

We run Monte Carlo simulation of the Spline-GTARCH model with 5000 observations generated with true parameters[2] given in the second column of Table 1. Figure 1 shows the results of fitting the Spline-GTARCH model for the high and low frequency volatilities. Table 1 shows the results of the estimated Spline-GTARCH, Spline-TARCH and Spline-GARCH parameters with 10 knots and most parameters are very close to true values. All GTARCH parameters are statistically significant, but not all the knots are significant. The best fitting among the three models is the Spline-GTARCH since it has the smallest Swartz information criterion. We also note that parameter δ is significant and shows higher GARCH persistence for negative returns compared to other two models.

The stationarity condition for the Spline-GTARCH model is given by: $\alpha+\beta+0.5\gamma+0.5\delta<1$. This condition generally holds for each model if we plug in zero values for parameters that are not in a model. From Table 1 we can see that all three models are stationary but the asymmetric Spline-GTARCH and Spline-TARCH models have higher overall persistence than the base Spline-GARCH model. The Monte Carlo simulations were also replicated 200 times and results were similar.[3]

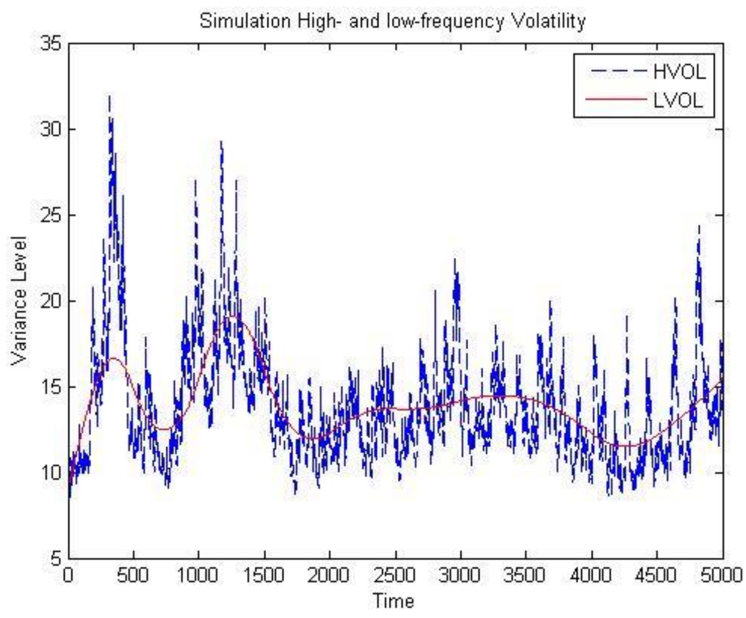

Figure 1. Simulation of S-GTARCH model

[2] The true parameters are the same as the results of estimation of the Spline-GTARCH model for DJI returns presented in the next section.

[3] Results are available from the authors on request.

Table 1. Numerical Example: Spline-GTARCH, Spline-TARCH and Spline-GARCH

Simulated SPLINE-GTARCH Nobs=5000, 9 knots							
		SPLINE-GTARCH		SPLINE-TARCH		SPLINE-GARCH	
	True parameters	coeff	stde	coeff	stde	coeff	stde
α	0.0222	0.0095	0.0084	0.0005	0.0076	**0.0727**	0.0076
β	**0.8785**	**0.8868**	0.0152	**0.9175**	0.0100	**0.8878**	0.0121
γ	**0.0778**	**0.0826**	0.0132	**0.1042**	0.0109		
δ	**0.0715**	**0.0616**	0.0197				
c	**0.8336**	**0.8557**	0.0796	**0.7863**	0.1053	**0.6352**	0.1030
w0	**8.0584**	59.7707	68.5185	158.6159	127.1882	**349.3839**	124.6291
w1	**-0.0031**	-0.0850	0.0992	-0.2218	0.1838	**-0.4975**	0.1757
w2	**-0.0042**	0.2048	0.1496	0.3879	0.2673	**0.8084**	0.2526
w3	**0.0286**	**-0.2982**	0.1130	**-0.3679**	0.1492	**-0.6243**	0.1475
w4	**-0.0387**	**0.2421**	0.1208	**0.3033**	0.1270	**0.5112**	0.1353
w5	0.0306	-0.0211	0.1210	-0.0955	0.1224	-0.2479	0.1406
w6	**-0.0344**	-0.0259	0.1207	0.0242	0.1199	0.0693	0.1481
w7	**0.0432**	-0.0825	0.1198	-0.0914	0.1089	-0.0464	0.1483
w8	**-0.0446**	0.0728	0.1128	0.0505	0.1046	-0.0112	0.1486
w9	**0.0529**	0.0224	0.1196	0.0656	0.1180	0.1321	0.1648
w10		0.0596	0.1861	-0.0019	0.1803	-0.0845	0.2372
α+β+0.5γ+0.5δ	0.9754	0.9685		0.9701		0.9605	
SIC		2.7287		2.7291		2.7453	

Notes: Data was simulated for Nobs=5000 using estimated coefficients of Spline-TGRACH model for DJI in Table3. Coefficients w0,w1,…w10 need to be multiplied E-05 to get actual values. Coefficients significant at 5% level are in bold.(α+β+0.5γ+0.5δ) measures persistence in the model. SIC is Swartz information Criterion.

4. Estimation Results for SPX and DJI

We use daily data for S&P 500 (SPX) and Dow Jones Industrial Average (DJI) for the period 1/3/1950-1/3/2013 from the Global Financial Database. There are 15,853 daily return observations[4]. The results of estimation of the three models for SPX are given in Table 2 and high and low frequency volatilities are given in Figures 2a-2c.

[4] Only trading days data were used.

Using Swartz information criterion the selected optimal number of knots is 8 for the Spline-GTARCH and 9 for the Spline-TARCH and Spline-GARCH[5]. All the GTARCH parameters are significant showing higher persistence of GARCH when returns are negative. Most of the spline parameters are statistically significant as well in the Spline-GTARCH and Spline-TARCH models. According to the Swartz Criterion the Spline-GTARCH model is preferred to other two models. All three models have high persistence, but satisfy stationarity requirement. Figures 2a-2c look similar except that the asymmetric models result in higher volatility peaks, such as October 1987.

Table 2. Estimation Results for SPX: Spline-GTARCH, Spline-TARCH and Spline-GARCH

	SGTARCH		STARCH		SGARCH	
Parameters	**coeff**	**stde**	**coeff**	**stde**	**coeff**	**stde**
α	**0.0286**	0.0035	**0.0197**	0.0027	**0.0867**	0.0045
β	**0.8642**	0.0077	**0.8995**	0.0043	**0.8944**	0.0057
γ	**0.0913**	0.0075	**0.1164**	0.0059		
δ	**0.0815**	0.0114				
c	**0.7955**	0.0718	**0.7719**	0.0776	**0.8108**	0.1101
w0	**63.1666**	13.3737	**48.6909**	16.1791	28.7324	23.1006
w1	**-0.0282**	0.0046	**-0.0213**	0.0061	-0.0135	0.0086
w2	**0.0434**	0.0070	**0.0210**	0.0088	0.0114	0.0124
w3	**-0.0145**	0.0049	**0.0219**	0.0060	**0.0221**	0.0088
w4	0.0017	0.0037	**-0.0369**	0.0048	**-0.0350**	0.0090
w5	**-0.0157**	0.0030	**0.0245**	0.0040	**0.0264**	0.0088
w6	**0.0267**	0.0032	**-0.0284**	0.0045	**-0.0336**	0.0076
w7	**-0.0258**	0.0034	**0.0420**	0.0045	**0.0484**	0.0065
w8	**0.0300**	0.0050	**-0.0455**	0.0056	**-0.0524**	0.0083
w9			**0.0553**	0.0102	**0.0607**	0.0142
Opt.knots	8		9		9	
$\alpha+\beta+0.5\gamma+0.5\delta$	0.9792		0.9774		0.9811	
SIC	2.4210		2.4236		2.4476	

Notes: Data for SPX 1/3/1950-1/3/2013 with 15,853 observations. Coefficients w0,w1,…w10 need to be multiplied E-05 to get actual values. Coefficients significant at 5% level are in bold.($\alpha+\beta+0.5\gamma+0.5\delta$) measures persistence in the model. SIC is the Swartz information Criterion.

[5] We set the maximum number of knots equal to 10.

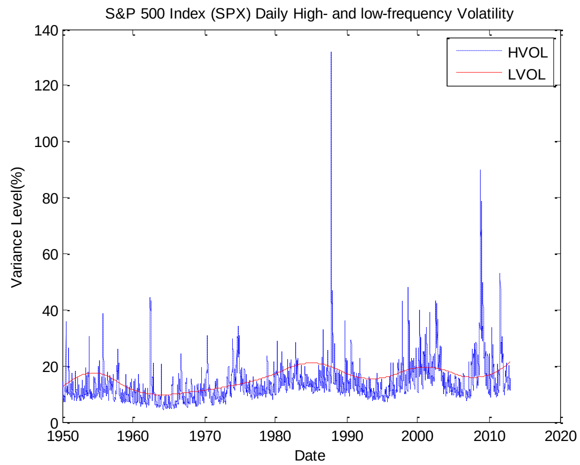

Figure 2a. Volatility Estimation using S-GTARCH Model

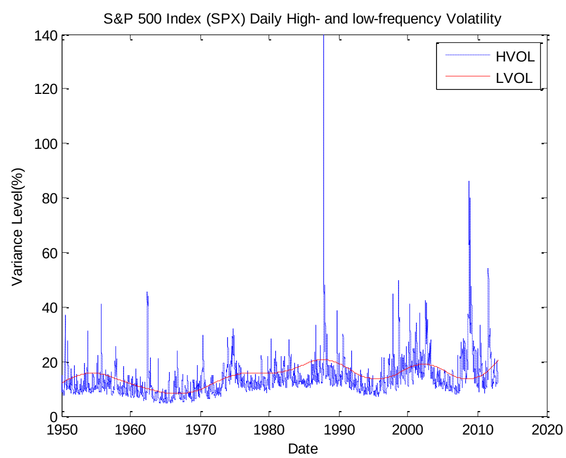

Figure 2b. S&P500 Estimation using S-TARCH model

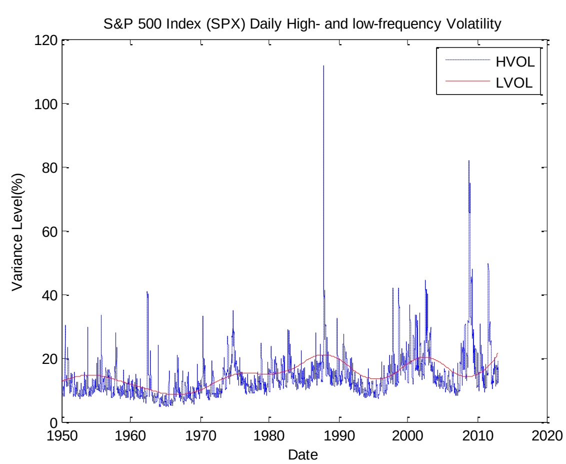

Figure 2c. S&P500 Estimation using S-GARCH model

The results of estimation of the three models for DJI are given in Table 3 and high and low frequency volatilities are given in Figures 3a-3c. The results are similar to the SPX. The preferred model is again Spline-GTARCH which exhibits higher persistence for the GARCH part when returns are negative. All three models select optimal number of knots equal to 9 and the results for the slow moving component are similar in Figures 3a-3c, while asymmetric models have higher peak for high-frequency component.

Table 3. Estimation Results for DJI: Spline-GTARCH, Spline-TARCH and Spline-GARCH

	SGTARCH		STARCH		SGARCH	
Parameters	**coeff**	**stde**	**coeff**	**stde**	**coeff**	**stde**
α	**0.0222**	0.0038	**0.0137**	0.0031	**0.0764**	0.0037
β	**0.8785**	0.0077	**0.9101**	0.0054	**0.9025**	0.0046
γ	**0.0778**	0.0066	**0.1045**	0.0062		
δ	**0.0715**	0.0112				
c	**0.8336**	0.0260	**0.6731**	0.0632	**0.7024**	0.0663
w0	**8.0584**	2.1637	**41.9552**	14.8379	**31.1653**	17.0666
w1	**-0.0031**	0.0017	**-0.0149**	0.0054	**-0.0109**	0.0065
w2	-0.0042	0.0034	**0.0108**	0.0070	0.0067	0.0097
w3	**0.0286**	0.0033	**0.0247**	0.0035	**0.0228**	0.0076
w4	**-0.0387**	0.0032	**-0.0386**	0.0036	**-0.0360**	0.0075
w5	**0.0306**	0.0032	**0.0319**	0.0038	**0.0333**	0.0069
w6	**-0.0344**	0.0032	**-0.0356**	0.0037	**-0.0402**	0.0058
w7	**0.0432**	0.0036	**0.0442**	0.0039	**0.0499**	0.0057
w8	**-0.0446**	0.0040	**-0.0455**	0.0048	**-0.0513**	0.0082
w9	**0.0529**	0.0061	**0.0549**	0.0071	**0.0602**	0.0139
Opt.knots	9		9		9	
α+β+0.5γ+0.5δ	0.9754		0.9761		0.9789	
SIC	2.4779		2.4798		2.5021	

Notes: Data for DJI 1/3/1950-1/3/2013 with 15,914 observations. Coefficients w0,w1,…w10 need to be multiplied E-05 to get actual values. Coefficients significant at 5% level are in bold.(α+β+0.5γ+0.5δ) measures persistence in the model. SIC is the Swartz information Criterion.

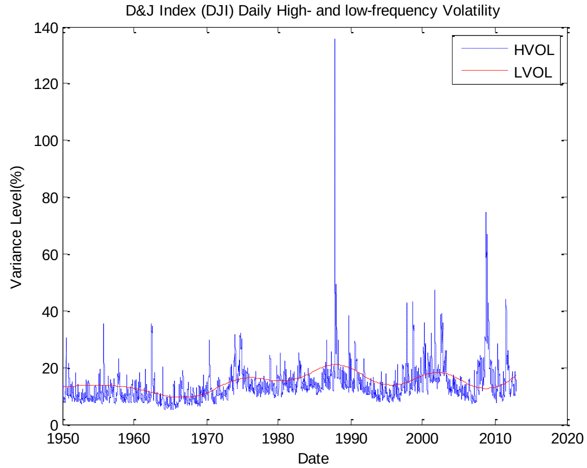

Figure 3a. D&J Estimation using S-GTARCH model

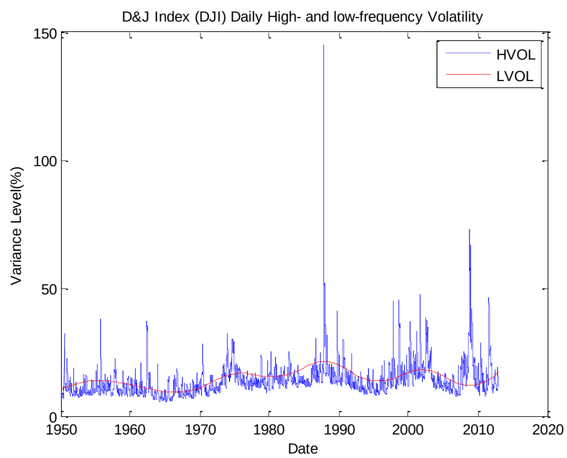

Figure 3b. D&J Estimation using S-TARCH model

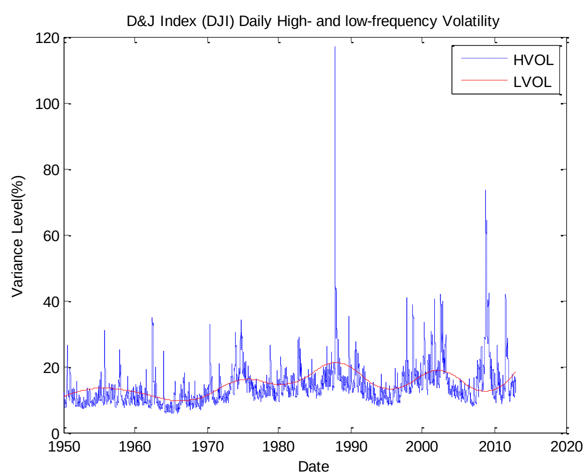

Figure 3c. D&J Estimation using S-GARCH model

4.1 Tail Risk

We start with the comparison of the multi-step-ahead volatility forecasts for three spline models. The forecast is made at the end of the sample which happens to be a period of relatively low volatility. The 1-day and 10-day volatility forecasts are given in the following Table.

	SGARCH	STARCH	SGTARCH
1 Day forecast	18.249%	14.792%	15.751%
10 Day forecast	18.808%	16.013%	16.855%

Volatility forecasts in all models are increasing but since it is a period of low volatility the numbers are smaller for Spline-TARCH and Spline-GTARCH. The Spline-GARCH model averages out TGARCH coefficients and therefore gives higher volatility values in a low-volatility regime.

Let us now compare the values of Value at Risk (VaR) and expected shortfall (ES) for the three models. For simplicity we use the assumption of Normal distribution for returns. This assumption can be easily relaxed and either t-distribution or bootstrap could be used.[6] The results for SPX data for various quantiles for VaR and ES are given in Tables 4 and 5 correspondingly.

Table 4. VaR for various quantiles of SPX

VaR	SGARCH	STARCH	SGTARCH
1-Day, p=90%	-1.474%	-1.195%	-1.272%
1-Day, p=95%	-1.891%	-1.533%	-1.632%
1-Day, p=99%	-2.674%	-2.167%	-2.308%
10-Day, p=90%	-1.519%	-1.293%	-1.361%
10-Day, p=95%	-1.949%	-1.659%	-1.747%
10-Day, p=99%	-2.756%	-2.346%	-2.470%

VaR is the (*1-p*) quantile of the distribution of returns, where p is the upper tail probability. In an extreme outcome, the actual loss (*L*) can be larger than VaR. In this case the actual expected loss in the tail is given by the expected shortfall ES:

$$ES_{1-p} = E(L \mid L > VaR_{1-p})$$

6 Although the results for VaR and ES will change with different distributional assumptions the ranking of the models will not change.

In case of the standard normal distribution,

$$ES_{1-p} = \frac{f(VaR_{1-p})}{p} \times \sigma_t$$

where σ_t is the volatility and f(x) is the standard normal density function.

Table 5. Expected Shortfall for various quantiles of SPX

ES	SGARCH	STARCH	SGTARCH
1-Day, p=90%	-2.018%	-1.635%	-1.741%
1-Day, p=95%	-2.372%	-1.922%	-2.047%
1-Day, p=99%	-3.064%	-2.483%	-2.644%
10-Day, p=90%	-2.079%	-1.770%	-1.863%
10-Day, p=95%	-2.444%	-2.081%	-2.190%
10-Day, p=99%	-3.158%	-2.688%	-2.830%

From Tables 4 and 5 we see that in the period of relatively low volatility Spline-GARCH model gives the highest values for tail risk, either VaR or ES. Surprisingly the Spline-TARCH model gives the lowest tail risk and the Spline-GTARCH models gives average results. As we noted from Figures 2 and 3 the Spline-TARCH model gives the highest volatility results when volatility is very high. The Spline-GTARCH model seems to be smoother and even though it accounts for asymmetry in the model it does not produce as extreme results as Spline-TARCH. Overall based on better fit and parameter significance we recommend using our model for forecasting volatility and measurement of tail risks as it is more general and robust than other two models. The computation of the likelihood function is straightforward and does not require more difficult estimation methods.

5. Conclusion

We introduced an asymmetric Spline-Threshold-GARCH model that generalizes the Spline-GARCH and asymmetric TARCH models. Monte Carlo experiments of this model and applications for SPX and DJI equity indices show that our model has better fit and has higher persistence for the GARCH parameters when the returns are negative. We suggest to use our model for forecasting volatility and tail risk measures as it is more general and robust than Spline-GARCH or Spline-TARCH models.

Our model can be extended for non-normal error distributions. Further research could be done on the economic determinants of low-frequency volatilities for the Spline-GTARCH model.

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